SIGNED q-ANALOGS OF TORNHEIM'S DOUBLE SERIES

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Abstract. We introduce signed q-analogs of Tornheim's double series, and evaluate them in terms of double q-Euler sums. As a consequence, we provide explicit evaluations of signed and unsigned Tornheim double series, and correct some mistakes in the literature.

1. Introduction

Let k be a positive integer. Sums of the form

$$\zeta(s_1, s_2, \dots, s_k) := \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{j=1}^{k} n_j^{-s_j}, \quad \sum_{j=1}^{m} \Re(s_j) > m, \quad m = 1, 2, \dots, k. \tag{1}$$

have attracted increasing attention in recent years; see eg. [2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 16, 24, 26, 27, 29, 30, 31]. The survey articles [7, 18, 19, 36, 37, 38] provide an extensive list of references. In (1) the sum is over all positive integers $n_1, n_2, \dots, n_k$ satisfying the indicated inequalities. Of course (1) reduces to the familiar Riemann zeta function when $k = 1$. When the arguments are all positive integers, we refer to (1) as a multiple zeta value and note that in this case, $s_1 > 1$ is necessary and sufficient for convergence.

The problem of evaluating $\zeta(s_1, s_2)$ with integers $s_1 > 1$ and $s_2 > 0$ seems to have been first proposed in a letter from Goldbach to Euler [21] in 1742. (See also [20, 22] and [1, p. 253].) Calculating several examples led Euler to infer a closed form evaluation in terms of values of the Riemann zeta function in the case when $s_1 + s_2$ is odd. In [4], Borwein, Bradley and Broadhurst considered the more general Euler sum

$$\zeta(s_1, s_2, \dots, s_k; \sigma_1, \sigma_2, \dots, \sigma_k) := \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{j=1}^{k} \sigma_j^{n_j} n_j^{-s_j} \tag{2}$$

with each $\sigma_j \in \{-1, 1\}$. Among the many other results for (2) listed therein is an explicit formula for the case $k = 2$ that reduces to Euler's evaluation when $\sigma_1 = \sigma_2 = 1$. For the case of arbitrary k, several infinite classes of closed form evaluations for (1) and (2) are

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proved in [5, 6, 8, 9, 10]. Subsequently, Bradley [12, 13, 14, 15] found q-analogs of (1) and (2) and thoroughly investigated their properties.

In [32] Tornheim introduced the double series

$$T(r,s,t) := \sum_{u,v=1}^{\infty} \frac{1}{u^r v^s (u+v)^t},$$

with nonnegative integers r, s and t satisfying $r+t>1$, $s+t>1$ and $r+s+t>2$. Huard, Williams and Zhang [25] evaluated Torheim's series in terms of sums of products of Riemann zeta values when $r+s+t$ is odd. Subbarao and Sitaramachandrarao [28] considered the alternating variants

$$R(r,s,t) := \sum_{u,v=1}^{\infty} \frac{(-1)^v}{u^r v^s (u+v)^t}, \qquad S(r,s,t) := \sum_{u,v=1}^{\infty} \frac{(-1)^{u+v}}{u^r v^s (u+v)^t},$$

and posed the problem to evaluate $R(r,r,r)$ and $S(r,r,r)$ for any positive integer r. Tsumura [33, Cor. 3] and [34, Theorem 3.6] tackled this problem and later [35] evaluated $S(r,s,t)$ for any positive integers r, s, t such that $r+s+t$ is odd. Tsumura's method is elementary but complicated, and also has some mistakes. In this paper, we give a simple formula that expresses R, S and T in terms of double Euler sums. In light of the aforementioned formula of Borwein, Bradley and Broadhurst for the double Euler sums, our results yield a closed form evaluation for the series R, S and T whenever the arguments r, s and t are positive integers with $r+s+t$ odd. More generally, we consider q-analogs of R, S, T and show how they may all be evaluated in terms of double q-Euler sums.

2. q-Analogs

Henceforth assume q is real and $q>1$. The q-analog of a positive integer n is

$$[n]_q := \sum_{j=0}^{n-1} q^j = \frac{q^n-1}{q-1}.$$

Let k be a positive integer, let $s_1, s_2, \dots, s_k$ be real numbers, and let $\sigma_1, \sigma_2, \dots, \sigma_k \in \{-1,1\}$. Define the q-Euler sum

$$\zeta_q[s_1, s_2, \dots, s_k; \sigma_1, \sigma_2, \dots, \sigma_k] := \sum_{n_1>n_2>\cdots>n_k>0} \prod_{j=1}^{k} \frac{\sigma_j^{n_j} q^{(s_j-1)n_j}}{[n_j]_q^{s_j}} \tag{3}$$

and note that this coincides with the special case $\mathrm{Li}_{s_1,\dots,s_k}[\sigma_1 q^{s_1-1}, \dots, \sigma_k q^{s_k-1}]$ of the multiple q-polylogarithm [12, eq. (6.2)]. If $\sigma_j = 1$ for each $j=1,2,\dots,k$, then we recover the multiple q-zeta value $\zeta[s_1, s_2, \dots, s_k]$ of [12, 14, 15].

Let $\sigma, \tau \in \{-1, 1\}$. Define q-analogs of the signed and unsigned Tornheim double series R, S and T by

$$T[r,s,t;\sigma,\tau] := \sum_{u,v=1}^{\infty} \frac{\sigma^u \tau^v q^{(r+t-1)u+(s+t-1)v}}{[u]_q^r [v]_q^s [u+v]_q^t}.$$

The sum

$$\varphi[s;\sigma] := \sum_{n=1}^{\infty} \frac{(n-1)\sigma^n q^{(s-1)n}}{[n]_q^s} = \sum_{n=1}^{\infty} \frac{n\sigma^n q^{(s-1)n}}{[n]_q^s} - \zeta_q[s;\sigma],$$

the case $\sigma = 1$ of which was defined in [15], will also be needed. As in [4], it is convenient to combine signs and exponents in (2) and (3) into a single list by writing s_j if $\sigma_j = 1$ and $\overline{s_j}$ if $\sigma_j = -1$. For consistency one may do this also for T and φ; thus for example, $\varphi[s;1] = \varphi[s]$, $\varphi[s;-1] = \varphi[\,\overline{s}\,]$, $R(r,s,t) = T(r,\overline{s},t)$ and $S(r,s,t) = T(\overline{r},\overline{s},t)$. We also employ the notation

$$\binom{z}{a,\ b} := \binom{z}{a}\binom{z-a}{b} = \binom{z}{b}\binom{z-b}{a}$$

for the trinomial coefficient, in which a, b are nonnegative integers, and which reduces to $z!/a!b!(z-a-b)!$ if z is also an integer exceeding $a+b$.

The following theorem shows how the q-analogs of R, S and T are related to the q-Euler sums.

Theorem 1. *Let r and s be positive integers, and let t be a real number. Then*

$$\begin{aligned}
T[r,s,t] &= \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a} \binom{a+s-1}{a,\ b}(1-q)^b \zeta_q[s+t+a, r-a-b] \\
&\quad + \sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a} \binom{a+r-1}{a,\ b}(1-q)^b \zeta_q[r+t+a, s-a-b] \\
&\quad - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j,\ s-j}(1-q)^j \varphi[r+s+t-j],
\end{aligned}$$

$$\begin{aligned}
T[\overline{r},\overline{s},t] &= \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a} \binom{a+s-1}{a,\ b}(1-q)^b \zeta_q[\,\overline{s+t+a}, r-a-b] \\
&\quad + \sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a} \binom{a+r-1}{a,\ b}(1-q)^b \zeta_q[\,\overline{r+t+a}, s-a-b] \\
&\quad - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j,\ s-j}(1-q)^j \varphi[\,\overline{r+s+t-j}\,],
\end{aligned}$$

$$\begin{aligned}T[r,\overline{s},t] &= \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\,b}(1-q)^b\zeta_q[\,\overline{s+t+a},\overline{r-a-b}\,]\\ &+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\,b}(1-q)^b\zeta_q[\,r+t+a,\overline{s-a-b}\,]\\ &-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\,s-j}(1-q)^j(1+q)^{j-r-s-t}\zeta_{q^2}[r+s+t-j].\end{aligned}$$

Taking the limit as $q\to 1+$ in Theorem 1 and noting the restrictions on r, s and t now needed for convergence yields the following

Corollary 1. *Let r and s be positive integers and let t be a real number. If $r+t>1$ and $s+t>1$, then*

$$T(r,s,t)=\sum_{a=0}^{r-1}\binom{a+s-1}{s-1}\zeta(s+t+a,r-a)+\sum_{a=0}^{s-1}\binom{a+r-1}{r-1}\zeta(r+t+a,s-a);$$

if $r+t>0$ and $s+t>0$, then

$$S(r,s,t)=\sum_{a=0}^{r-1}\binom{a+s-1}{s-1}\zeta(\overline{s+t+a},r-a)+\sum_{a=0}^{s-1}\binom{a+r-1}{r-1}\zeta(\overline{r+t+a},s-a);$$

if $r+t>1$ and $s+t>0$, then

$$R(r,s,t)=\sum_{a=0}^{r-1}\binom{a+s-1}{s-1}\zeta(\overline{s+t+a},\overline{r-a})+\sum_{a=0}^{s-1}\binom{a+r-1}{r-1}\zeta(r+t+a,\overline{s-a}).$$

Putting $t=0$ in Theorem 1 yields the following decomposition formulas, the first of which was given under slightly more restrictive hypotheses in [15, Theorem 2.1].

Corollary 2. *If r and s are positive integers, then*

$$\begin{aligned}\zeta_q[r]\zeta_q[s] &= \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{s-1}\binom{s-1}{b}(1-q)^b\zeta_q[s+a,r-a-b]\\ &+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{r-1}\binom{r-1}{b}(1-q)^b\zeta_q[r+a,s-a-b]\\ &-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\,s-j}(1-q)^j\varphi[r+s-j];\end{aligned}$$

$$
\begin{aligned}
\zeta_q[\,\overline{r}\,]\zeta_q[\,\overline{s}\,] &= \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{s-1}\binom{s-1}{b}(1-q)^b\zeta_q[\,\overline{s+a},r-a-b] \\
&+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{r-1}\binom{r-1}{b}(1-q)^b\zeta_q[\,\overline{r+a},s-a-b] \\
&-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\ s-j}(1-q)^j\varphi[\,\overline{r+s-j}\,]; \\
\zeta_q[r]\zeta_q[\,\overline{s}\,] &= \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{s-1}\binom{s-1}{b}(1-q)^b\zeta_q[\,\overline{s+a},\overline{r-a-b}\,] \\
&+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{r-1}\binom{r-1}{b}(1-q)^b\zeta_q[r+a,\overline{s-a-b}\,] \\
&-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\ s-j}(1-q)^j(1+q)^{j-r-s}\zeta_{q^2}[r+s-j].
\end{aligned}
$$

Taking the limit as $q \to 1+$ in Corollary 2 and noting the additional restrictions needed on r and s to guarantee convergence in this case yields the following decomposition formulas, the first of which was known to Euler.

Corollary 3. *If $r-1$ and $s-1$ are positive integers, then*

$$\zeta(r)\zeta(s) = \sum_{a=0}^{r-1}\binom{a+s-1}{s-1}\zeta(s+a,r-a) + \sum_{a=0}^{s-1}\binom{a+r-1}{r-1}\zeta(r+a,s-a);$$

if r and s are positive integers, then

$$\zeta(\overline{r})\zeta(\overline{s}) = \sum_{a=0}^{r-1}\binom{a+s-1}{s-1}\zeta(\overline{s+a},r-a) + \sum_{a=0}^{s-1}\binom{a+r-1}{r-1}\zeta(\overline{r+a},s-a);$$

if $r-1$ and s are positive integers, then

$$\zeta(r)\zeta(\overline{s}) = \sum_{a=0}^{r-1}\binom{a+s-1}{s-1}\zeta(\overline{s+a},\overline{r-a}) + \sum_{a=0}^{s-1}\binom{a+r-1}{r-1}\zeta(r+a,\overline{s-a}).$$

3. Proof of Theorem 1.

The key ingredient is the following partial fraction decomposition.

Lemma 1. *If r, s, u, and v are all positive integers, then*

$$\begin{aligned}\frac{1}{[u]_q^r[v]_q^s} &= \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\,b}\frac{(1-q)^b\,q^{(s-1-b)u+av}}{[u]_q^{r-a-b}[u+v]_q^{s+a}}\\ &\quad+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\,b}\frac{(1-q)^b\,q^{au+(r-1-b)v}}{[v]_q^{s-a-b}[u+v]_q^{r+a}}\\ &\quad-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\,s-j}\frac{(1-q)^j\,q^{(s-j)u+(r-j)v}}{[u+v]_q^{r+s-j}}.\end{aligned}$$

Proof. Let x and y be nonzero real numbers such that $x+y+(q-1)xy\neq 0$. As in [15], observe that if we apply the partial differential operator

$$\frac{1}{(r-1)!}\bigg(-\frac{\partial}{\partial x}\bigg)^{r-1}\frac{1}{(s-1)!}\bigg(-\frac{\partial}{\partial y}\bigg)^{s-1}$$

to both sides of the identity

$$\frac{1}{xy}=\frac{1}{x+y+(q-1)xy}\bigg(\frac{1}{x}+\frac{1}{y}+q-1\bigg),$$

then we obtain the identity [15, Lemma 3.1]

$$\begin{aligned}\frac{1}{x^ry^s} &= \sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\,b}\frac{(1-q)^b(1+(q-1)y)^a(1+(q-1)x)^{s-1-b}}{x^{r-a-b}(x+y+(q-1)xy)^{s+a}}\\ &\quad+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\,b}\frac{(1-q)^b(1+(q-1)x)^a(1+(q-1)y)^{r-1-b}}{y^{s-a-b}(x+y+(q-1)xy)^{r+a}}\\ &\quad-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\,s-j}\frac{(1-q)^j\,(1+(q-1)y)^{r-j}(1+(q-1)x)^{s-j}}{(x+y+(q-1)xy)^{r+s-j}}.\end{aligned}$$

Now let $x=[u]_q$, $y=[v]_q$ and note that then $1+(q-1)x=q^u$, $1+(q-1)y=q^v$ and $x+y+(q-1)xy=[u+v]_q$. □

To prove Theorem 1, multiply both sides of Lemma 1 by

$$\frac{\sigma^u\tau^v q^{(r+t-1)u+(s+t-1)v}}{[u+v]_q^t}$$

and sum over all ordered pairs of positive integers (u,v) to obtain

$$T[r,s,t;\sigma,\tau]=\sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\,b}(1-q)^b\sum_{u,v=1}^{\infty}\frac{\sigma^u\tau^v q^{(r-a-b-1)u}q^{(s+t+a-1)(u+v)}}{[u]_q^{r-a-b}[u+v]_q^{s+t+a}}$$

$$+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\ b}(1-q)^b\sum_{u,v=1}^{\infty}\frac{\sigma^u\tau^v q^{(s-a-b-1)v}q^{(r+t+a-1)(u+v)}}{[v]_q^{s-a-b}[u+v]_q^{r+t+a}}$$
$$-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\ s-j}(1-q)^j\sum_{u,v=1}^{\infty}\frac{\sigma^u\tau^v q^{(r+s+t-j-1)(u+v)}}{[u+v]_q^{r+s+t-j}}.$$

It follows that

$$\begin{aligned}T[r,s,t;\sigma,\sigma]&=\sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\ b}(1-q)^b\sum_{m>n>0}\frac{\sigma^m q^{(s+t+a-1)m}q^{(r-a-b-1)n}}{[m]_q^{s+t+a}[n]_q^{r-a-b}}\\&+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\ b}(1-q)^b\sum_{m>n>0}\frac{\sigma^m q^{(r+t+a-1)m}q^{(s-a-b-1)n}}{[m]_q^{r+t+a}[n]_q^{s-a-b}}\\&-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\ s-j}(1-q)^j\sum_{m>n>0}\frac{\sigma^m q^{(r+s+t-j-1)m}}{[m]_q^{r+s+t-j}}\\&=\sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\ b}(1-q)^b\zeta_q[s+t+a,r-a-b;\sigma,1]\\&+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\ b}(1-q)^b\zeta_q[r+t+a,s-a-b;\sigma,1]\\&-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\ s-j}(1-q)^j\varphi[r+s+t-j;\sigma],\end{aligned}$$

and also that

$$\begin{aligned}T[r,s,t;1,-1]&=\sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\ b}(1-q)^b\sum_{m>n>0}\frac{(-1)^m q^{(s+t+a-1)m}(-1)^n q^{(r-a-b-1)n}}{[m]_q^{s+t+a}[n]_q^{r-a-b}}\\&+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\ b}(1-q)^b\sum_{m>n>0}\frac{q^{(r+t+a-1)m}(-1)^n q^{(s-a-b-1)n}}{[m]_q^{r+t+a}[n]_q^{s-a-b}}\\&-\sum_{j=1}^{\min(r,s)}\binom{r+s-j-1}{r-j,\ s-j}(1-q)^j\sum_{m>n>0}\frac{(-1)^n q^{(r+s+t-j-1)m}}{[m]_q^{r+s+t-j}}\\&=\sum_{a=0}^{r-1}\sum_{b=0}^{r-1-a}\binom{a+s-1}{a,\ b}(1-q)^b\zeta_q[s+t+a,r-a-b;\sigma,1]\\&+\sum_{a=0}^{s-1}\sum_{b=0}^{s-1-a}\binom{a+r-1}{a,\ b}(1-q)^b\zeta_q[r+t+a,s-a-b;\sigma,1]\end{aligned}$$

$$+\sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j,\ s-j}(1-q)^j \sum_{m>0} \frac{q^{(r+s+t-j-1)2m}}{[2m]_q^{r+s+t-j}}.$$

Since

$$[2m]_q = \frac{q^{2m}-1}{q-1} = \frac{q^2-1}{q-1}\cdot\frac{q^{2m}-1}{q^2-1} = (q+1)[m]_{q^2},$$

the proof of Theorem 1 is complete. $\square$

4. Examples

Again, let $\sigma,\tau \in \{-1,1\}$. It is known [4, 23] that if s and t are positive integers such that $s+t$ is odd and $s>(1+\sigma)/2$, then $\zeta(s,t;\sigma,\tau)$ lies in the polynomial ring $\mathbf{Q}[\{\zeta(k;\pm1) : k\in\mathbf{Z}, 2\le k\le s+t\}\cup\{\zeta(1;-1)\}]$. Since

$$\zeta(k;-1) = \zeta(\overline{k}) = \begin{cases} \big(2^{1-k}-1\big)\zeta(k) & \text{if } k>1,\\ -\log 2 & \text{if } k=1,\end{cases}$$

it is clear that in fact, $\zeta(s,t;\sigma,\tau)\in\mathbf{Q}[\{\zeta(k) : k\in\mathbf{Z}, 2\le k\le s+t\}\cup\{-\log 2\}]$. It follows that if r,s,t satisfy the conditions of Corollary 1 and if in addition $r+s+t$ is an odd integer, then the signed and unsigned double Tornheim series $R(r,s,t)$, $S(r,s,t)$ and $T(r,s,t)$ also lie in this ring. It is possible to evaluate these series explicitly if we recall the following formula from [4, eq. (75)].

Proposition 1. *Let $\sigma,\tau\in\{-1,1\}$, and let s and t be positive integers such that $s+t$ is odd, $s>(1+\sigma)/2$, and $t>(1+\tau)/2$. Then*

$$\begin{aligned}\zeta(s,t;\sigma,\tau) &= \tfrac12\big(1+(-1)^s\big)\zeta(s;\sigma)\zeta(t;\tau) - \tfrac12\zeta(s+t;\sigma\tau)\\ &\quad + (-1)^t \sum_{0\le k\le t/2}\binom{s+t-2k-1}{s-1}\zeta(2k;\sigma\tau)\zeta(s+t-2k;\sigma)\\ &\quad + (-1)^t \sum_{0\le k\le s/2}\binom{s+t-2k-1}{t-1}\zeta(2k;\sigma\tau)\zeta(s+t-2k;\tau). \end{aligned}\tag{4}$$

In Proposition 1, it is understood that $\zeta(0;\sigma\tau)=-1/2$ in accordance with the analytic continuation of $s\mapsto\zeta(s;\sigma\tau)$. The restriction $t>(1+\tau)/2$ can be removed if in (4) we interpret $\zeta(1;1)=0$ wherever it occurs. That is, if $\sigma\in\{-1,1\}$ and s is an even positive integer, then

$$\zeta(s,1;\sigma,1) = \frac12(s-1)\zeta(s+1;\sigma) + \frac12\zeta(s+1) - \sum_{k=1}^{(s/2)-1}\zeta(2k;\sigma)\zeta(s+1-2k). \tag{5}$$

The case $\sigma = 1$ of (5) is subsumed by another formula [4, eq. (31)] of Euler, namely

$$\zeta(s,1) = \frac{s}{2}\,\zeta(s+1) - \frac{1}{2}\sum_{k=2}^{s-1}\zeta(k)\zeta(s+1-k), \tag{6}$$

which is valid for *all* integers $s > 1$, not just for even s.

In [35], Tsumura listed evaluation formulas for $S(r,s,t)$ when $r+s+t \le 9$ is odd. From Corollary 1, Proposition 1 and equation (5), we can deduce explicit formulas for $R(r,s,t)$, $S(r,s,t)$ and $T(r,s,t)$ when $r+s+t$ is odd. In particular, we have the following new results:

$$R(1,1,1) = -\frac{5}{8}\zeta(3), \qquad R(1,1,3) = \frac{1}{16}\pi^2\zeta(3) - \frac{27}{32}\zeta(5),$$
$$R(1,2,2) = \frac{5}{48}\pi^2\zeta(3) - \frac{3}{2}\zeta(5), \qquad R(1,3,1) = \frac{1}{12}\pi^2\zeta(3) - \frac{59}{32}\zeta(5),$$
$$R(2,1,2) = -\frac{5}{16}\pi^2\zeta(3) + \frac{107}{32}\zeta(5), \qquad R(2,2,1) = -\frac{5}{24}\pi^2\zeta(3) + \frac{59}{32}\zeta(5),$$
$$R(3,1,1) = \frac{1}{8}\pi^2\zeta(3) - \frac{59}{32}\zeta(5),$$

$$S(5,5,5) = \frac{7}{73728}\pi^4\zeta(11) + \frac{35}{24576}\pi^2\zeta(13) + \frac{63}{8192}\zeta(15),$$
$$S(7,7,7) = \frac{31}{35389440}\pi^6\zeta(15) + \frac{49}{1966080}\pi^4\zeta(17) + \frac{77}{262144}\pi^2\zeta(19) + \frac{429}{262144}\zeta(21).$$

The values of $R(5,5,5)$, $R(7,7,7)$ and $R(9,9,9)$ listed in [34] appear to be incorrect. They should be

$$R(5,5,5) = \frac{16375}{147456}\pi^4\zeta(11) + \frac{573335}{49152}\pi^2\zeta(13) - \frac{2064195}{16384}\zeta(15),$$
$$R(7,7,7) = \frac{1048543}{70778880}\pi^6\zeta(15) + \frac{7339969}{3932160}\pi^4\zeta(17) + \frac{80740121}{524288}\pi^2\zeta(19) - \frac{899676921}{524288}\zeta(21),$$
$$\begin{aligned} R(9,9,9) = {} & \frac{13421747}{7046430720}\pi^8\zeta(19) + \frac{738197141}{2113929216}\pi^6\zeta(21) + \frac{1919313253}{67108864}\pi^4\zeta(23) \\ & + \frac{143948506845}{67108864}\pi^2\zeta(25) - \frac{1631416447375}{67108864}\zeta(27). \end{aligned}$$

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